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Another Case of Nonexistence in Rational Chebyshev Approximation with Interpolatory Constraints

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1. INTRODUCTION

Let $I=[0,1]$ and for nonnegative integers n and m let $R_{n,m}=\{r=p/q\colon p\in\prod_n,\ q\in\prod_m,$ and $q>0$ on $I\}$. Here $\prod_l$ denotes the space of real algebraic polynomials of degree l or less. The problem of rational Chebyshev approximation with Lagrange interpolatory constraints is described as follows. We fix s points $0\leqslant t_1<\cdots<t_s\leqslant 1$ and for $f\in C(I)$ we seek r^* in $R_{n,m}(f)=\{r\in R_{n,m}\colon r(t_i)=f(t_i)\ \ (i=1,...,s)\}$ such that $\|f-r^*\|=\inf\{\|f-r\|\colon r\in R_{n,m}(f)\}$ where $\|\cdot\|$ denotes the uniform norm over I. We call such an r^* a best approximation to f from $R_{n,m}(f)$. It has long been known that there is a difficulty with existence for this problem. When $n\geqslant s-1$, so that $R_{n,m}(f)\neq\varnothing$ for all $f\in C(I)$, Gilormini [4] announced a positive existence result for this problem along with a characterization theorem. However, Loeb [5] disproved the existence assertion giving an example of a function $f\in C(I)$ that fails to have a best approximation from $R_{1,1}(f)$ when $s=1$ and $t_1=0$. In Loeb's example, f is not normal with respect to $R_{1,1}$. (We say that $f\in C(I)$ is normal with respect to $R_{n,m}$ if its unique best approximation from $R_{n,m}$ is in $R_{n,m}\backslash R_{n-1,m-1}$.) More recently, when $s=1$, Dunham [3] gave conditions that ensure that a function that is not normal with respect to $R_{n,m}$ fails to have a best approximation from $R_{n,m}(f)$. He also announced an example of a normal function for which existence for the constrained problem fails. It is of interest to determine the extent of this nonexistence phenomenon. Something analogous to a result of Cheney and Loeb [2] that the set of functions in $C(I)$ that are not normal with respect to $R_{n,m}$ is nowhere dense in $C(I)$ would be desirable. To the contrary, however, the principle result of this short note is that the nonexistence phenomenon is quite rampant.

For simplicity, we only consider the case $s=1$ and $t_1=0$ so that $R_{n,m}(f)=\{r\in R_{n,m}\colon r(0)=f(0)\}$. When $n, m\geqslant 1$, we give a condition based

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on the best approximation to f from $R_{n-1,m-1}$ which implies that f has no best approximation from $R_{n,m}(f)$. We then use this result and the continuity properties of the best approximation operator corresponding to $R_{n-1,m-1}$ to demonstrate a nonempty, open subset of $C(I)$ all of whose elements fail to have best approximations from $R_{n,m}$ with the interpolation constraint. A consequence of this result is that for fixed $f \in C(I)$, n and m being sufficiently large is not enough to ensure existence for the constrained problem.

2. Nonexistence Results

For $f \in C(I)$ and $S \subseteq C(I)$, let $\operatorname{dist}(f, S) = \inf\{\|f-g\|: g \in S\}$. We use the following lemma.

Lemma. *Let $n, m \geqslant 1$ and $f \in C(I)$. Then*

$$\operatorname{dist}(f, R_{n,m}) \leqslant \operatorname{dist}(f, R_{n,m}(f)) \leqslant \operatorname{dist}(f, R_{n-1,m-1}).$$

Proof. The first inequality is trivial. For the second inequality, let $r = p/q$ be the best approximation to f from $R_{n-1,m-1}$ where $p \in \prod_{n-1}$, $q \in \prod_{m-1}$, and $q > 0$ on I. If $\|f-r\| = 0$, then $r(0) = f(0)$ and $\operatorname{dist}(f, R_{n,m}(f)) = \operatorname{dist}(f, R_{n-1,m-1}) = 0$. Assume $\|f-r\| > 0$. For k a positive integer, define

$$r_k(x) = (xp(x) + f(0)/k)/(xq(x) + 1/k).$$

Note that $r_k \in R_{n,m}(f)$ and that

$$f(x) - r_k(x) = \frac{xq(x)}{xq(x)+1/k}(f(x) - r(x)) + \frac{1/k}{xq(x)+1/k}(f(x) - f(0)).$$

Hence, for $x \in I$

$$|f(x) - r_k(x)| \leqslant \frac{xq(x)}{xq(x)+1/k}\|f-r\| + \frac{1/k}{xq(x)+1/k}|f(x) - f(0)|. \quad (*)$$

Choose $\delta > 0$ so that $|f(x) - f(0)| \leqslant \|f-r\|$ for $x \in [0, \delta]$. Clearly, $(*)$ implies that $|f(x) - r_k(x)| \leqslant \|f-r\|$ for $x \in [0, \delta]$ and that $|f-r_k| \to |f-r|$ uniformly on $[\delta, 1]$. Hence

$$\limsup_{k\to\infty} \|f-r_k\| \leqslant \|f-r\|$$

and so $\operatorname{dist}(f, R_{n,m}(f)) \leqslant \operatorname{dist}(f, R_{n-1,m-1})$.

For $f \in C(I)$, let $E(f) = \{x \in I: |f(x)| = \|f\|\}$. Evidently, $E(f)$ is compact in I.

THEOREM 1. *Let $n, m \geqslant 1$, $f \in C(I)$, r be the best approximation to f from $R_{n-1,m-1}$, and let x_1 be the smallest element of $E(f-r)$. If $x_1 > 0$ and $(f(0) - r(0))\,(f(x_1) - r(x_1)) < 0$, then f does not have a best approximation from $R_{n,m}(f)$.*

Proof. Let $r = p/q$ (in reduced form) where $p \in \prod_{n-1}$, $q \in \prod_{m-1}$, and $q > 0$ on I, and let $d = \min(n - \deg p,\ m - \deg q)$. By the alternation theorem for rational approximation (see [1, 2]) $f - r$ exhibits at least $l + 1 \equiv n + m + 1 - d$ points of alternation in $E(f - r)$. That is, there exist points $x_1 < \cdots < x_{l+1}$ in I where $(f - r)(x_i) = \sigma(-1)^i \,\|f - r\|$ $(i = 1, ..., l + 1)$ where $\sigma = \pm 1$. Since x_1 is the smallest element of $E(f - r)$, we can choose x_1 to be the first point in the alternant.

Assume that f has a best approximation $r^* = p^*/q^*$ from $R_{n,m}(f)$ whence $p^* \in \prod_n$, $q^* \in \prod_m$, $q^* > 0$ on I, and $r^*(0) = f(0)$. By hypothesis, $r^*(0) \neq r(0)$ so that $r^* \neq r$. By the lemma, $\|f - r^*\| \leqslant \|f - r\|$ and thus for $i = 1, ..., l + 1$,

$$\begin{aligned}\sigma(-1)^i\,(f - r^*)(x_i) &\leqslant \|f - r^*\| \\ &\leqslant \|f - r\| = \sigma(-1)^i (f - r)(x_i)\end{aligned}$$

so that

$$\sigma(-1)^i\,(r^* - r)(x_i) \geqslant 0.$$

Since $-\sigma = \operatorname{sgn}(f - r)(x_1)$, the hypothesis yields

$$\sigma(-1)^0\,(r^* - r)(x_0) = \sigma(f - r)(0) > 0,$$

where $x_0 = 0$. Since $q, q^* > 0$ on I, $\sigma(-1)^i\,(p^*q - q^*p)(x_i) \geqslant 0$ $(i = 0, ..., l + 1)$. But $p^*q - q^*p \in \prod_l$ and thus $p^*q - q^*p \equiv 0$ so that $r^* = r$, a contradiction. Theorem 1 is now proven.

If $f \in C(I)$ is normal with respect to $R_{n-1,m-1}$, then the alternation theorem implies that $E(f - r)$ contains at least $n + m$ points where r is the best approximation to f from $R_{n-1,m-1}$. In Theorem 1, if f is normal and $E(f - r)$ contains precisely $n + m$ points, then all functions sufficiently near f fail to have best approximations from $R_{n,m}$ with the interpolatory constraint.

THEOREM 2. *Let $f \in C(I)$ satisfy the conditions of Theorem 1, and further suppose that f is normal with respect to $R_{n-1,m-1}$ and that $E(f - r)$ contains precisely $n + m$ points. Then there exists $\varepsilon > 0$ such that for every $g \in C(I)$ with $\|g - f\| < \varepsilon$, g does not have a best approximation from $R_{n,m}(g)$.*

Proof. Let $l = n + m$ and $E(f - r) = \{x_1, ..., x_l\}$ where $0 < x_1 < \cdots < x_l \leqslant 1$. Assume the conclusion is false. Then there is a sequence (g_k) in $C(I)$ such that $\|g_k - f\| \to 0$ and each g_k has a best approximation from $R_{n,m}(g_k)$. Let r_k denote the best approximation to g_k from $R_{n-1,m-1}$ and let ξ_1^k be the smallest element of $E(g_k - r_k)$. By the continuity of the best approximation operator at each normal function [6], $\|r_k - r\| \to 0$.

Moreover, the set of functions that are normal with respect to $R_{n-1,m-1}$ is open in $C(I)$ (see [2]). Thus for k sufficiently large, g_k is normal with respect to $R_{n-1,m-1}$ and we may then choose an alternant $\xi_1^k < \cdots < \xi_l^k$ for $g_k - r_k$ consisting of l points in $E(g_k - r_k)$. We now extract a subsequence and relabel so that each g_k is normal with respect to $R_{n-1,m-1}$ and $\xi_i^k \to \xi_i$ $(i = 1, ..., l)$ where $\xi_1 \leqslant \cdots \leqslant \xi_l$. The convergent sequence $(g_k - r_k)$ is precompact in $C(I)$ and thus is equicontinuous. Hence,

$$|(f-r)(\xi_i)| = \lim_{k \to \infty} |(g_k - r_k)(\xi_i^k)|$$
$$= \lim_{k \to \infty} \|g_k - r_k\| = \|f - r\|$$

$(i = 1, ..., l)$. So each $\xi_i \in E(f - r)$. But for $i = 1, ..., l - 1$,

$$(f-r)(\xi_i)(f-r)(\xi_{i+1}) = \lim (g_k - r_k)(\xi_i^k)(g_k - r_k)(\xi_{i+1}^k)$$
$$= -\|f - r\|^2 < 0.$$

Thus, $\xi_1 < \cdots < \xi_l$, and so, $\xi_i = x_i$ $(i = 1, ..., l)$. Finally,

$$\lim_{k \to \infty} (g_k - r_k)(0)(g_k - r_k)(\xi_1^k) = (f - r)(0)(f - r)(x_1) < 0.$$

So for k sufficiently large, $\xi_1^k > 0$ and $(g_k - r_k)(0)(g_k - r_k)(\xi_1^k) < 0$ and by Theorem 1, g_k has no best approximation fom $R_{n,m}(g_k)$. We have a contradiction and Theorem 2 is proven.

We point out that functions $f \in C(I)$ satisfying the conditions of Theorem 2 are easy to come by. Start with $r \in R_{n-1,m-1} \setminus R_{n-2,m-2}$ and let $f = r + h$ where $h \in C(I)$, $|h| < \rho$ on $I \setminus \{x_1, ..., x_{n+m}\}$ where $0 < x_1 < \cdots < x_{n+m} \leqslant 1$ are arbitrary, $h(0) > 0$, and $h(x_i) = (-1)^i \rho$ $(i = 1, ..., n + m)$ where $\rho > 0$. Thus Theorem 2 indeed demonstrates nonempty open subsets of $C(I)$ over which existence for the constrained problem fails.

We now turn our attention to the existence question with f fixed and n and m varying. A consequence of the construction above is that for fixed $f \in C(I)$ existence of a best approximation from $R_{n,m}(f)$ is not guaranteed for n and m sufficiently large.

THEOREM 3. *Let $(n(k))$ and $(m(k))$ be sequences of positive integers with $n(k) \to \infty$ as $k \to \infty$. Then there exists $f \in C(I)$ such that for infinitely many k, f does not have a best approximation from $R_{n(k),m(k)}(f)$.*

Proof. Let $A_k = \{g \in C(I): g$ has a best approximation from $R_{n(k),m(k)}(g)\}$ and $F_k = \bigcap_{j \geqslant k} A_j$. Evidently F_k is closed, and we show that each F_k has an empty interior in $C(I)$. Let $g \in F_k$ and $\varepsilon > 0$. By the Weierstrass theorem choose a polynomial p so that $\|p - g\| < \varepsilon/2$. Now choose $j \geqslant k$ so that $\deg p \leqslant n(j) - 1$. By adding a suitably small multiple of $x^{n(j)-1}$ to $p(x)$, we may assume that $\deg p = n(j) - 1$. Then $p \in R_{n(j)-1, m(j)-1} \backslash R_{n(j)-2, m(j)-2}$, and the construction above with $r = p$, $n = n(j)$, $m = m(j)$, and $\rho = \varepsilon/2$ yields $f \notin A_j$ with $\|f - g\| < \varepsilon$. Hence, F_k has an empty interior. By the Baire category theorem, $C(I)$ contains some f not in $\bigcup_{k \geqslant 1} F_k$. In particular, for each k, f has no best approximation from $R_{n(j),m(j)}(f)$ for some $j \geqslant k$. The proof is complete.

We conclude this note by mentioning that one can obtain sufficient conditions for $f \in C(I)$ to have a best approximation from $R_{n,m}(f)$. Such conditions include $\operatorname{dist}(f, R_{n,m}(f)) < \operatorname{dist}(f, R_{n-1,m-1})$ or $(f - r_0)(0)(f - r_1)(0) < 0$ where r_i denotes the best approximation to f from $R_{n-i,m-i}$ $(i = 0, 1)$. Proofs are similar to that of the theorem on p. 155 in [1] with an additional argument preventing cancellations.

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